— The World's Best Way to Complete the Square —
(According to Robert)

Background – We use this technique to change the form of quadratic expressions so that we can more clearly "see" the behavior.

The key idea relates to a binomial square

$$a^2+2ab+b^2=(a+b)^2 \;\&\; a^2-2ab+b^2=(a-b)^2$$

Example – Follow each step

Let's find the "vertex" and "zeros" (x-intercepts) of

$$y=3x^2-5x-7$$

Step #1: Divide both sides by the leading coefficient.

$$\frac{y}{3}=x^2-\frac{5}{3}x-\frac{7}{3}$$

Step #2: Take $\frac{1}{2}$ of the linear coefficient, write it down & circle it.

$$\frac{y}{3}=x^2-\frac{5}{3}x-\frac{7}{3}$$

$$\frac{1}{2}\left(\frac{-5}{3}\right)=\frac{-5}{6}$$

Step #3: Square it and insert it as shown.

$$\frac{y}{3}=x^2-\frac{5}{3}x+\frac{25}{36}-\frac{7}{3}-\frac{25}{36}$$

Step #4: The first three terms form a binomial square & the last numbers add together.

$$\frac{y}{3}=\left(x-\frac{5}{6}\right)^2-\frac{7\cdot 12}{3\cdot 12}-\frac{25}{36}$$

↑ see circled value from Step #2.

$$\frac{y}{3}=\left(x-\frac{5}{6}\right)^2-\frac{109}{36}$$

Step #5: Multiply both sides by 3.

$$\boxed{y=3\left(x-\frac{5}{6}\right)^2-\frac{109}{12}}$$

Now, Use results from Step #5.

The vertex is $\left(\frac{5}{6},-\frac{109}{12}\right)$ and the parabola opens upward since $3>0$!!

To find the zeros take the result of Step #4 & let $y=0$!! So,

$$\left(x-\frac{5}{6}\right)^2-\frac{109}{36}=0$$

Solve for x,

$$\left(x-\frac{5}{6}\right)^2=\frac{109}{36}$$

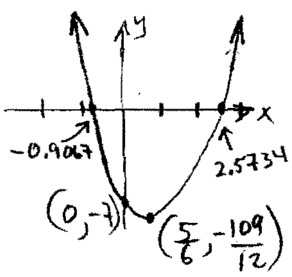
$$x-\frac{5}{6}=\frac{\pm\sqrt{109}}{6}$$

$$x=\frac{5\pm\sqrt{109}}{6}$$

Using a calculator, approximate values are:

$$x_1\approx 2.5734$$

$$x_2\approx -0.9067$$



— Completing the Square to locate the Vertex of Any quadratic function and to determine its zeros.— (The quadratic formula)

Given: $y = ax^2 + bx + c$

This approach also can be used if numbers are given for a, b, & c. (See other handout)

Divide both sides by "a":

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Take $\frac{1}{2}$ of the linear coefficient, circle it, square it, and insert as shown

$$\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}$$

$$\frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a}$$

Factor into the "binomial square" and combine the last two terms using their LCD.

$$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a}\left(\frac{4a}{4a}\right) - \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

Multiply both sides by "a":

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

If $x = -\frac{b}{2a}$, then $y = \frac{4ac - b^2}{4a}$ which is the "Vertex" $\boxed{V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)}$

Now, let $y = 0$ & solve for x (follow the steps)

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

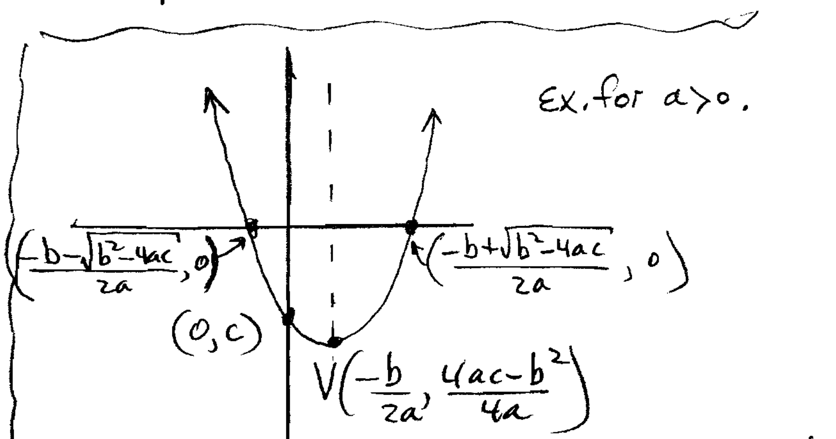
$$\left(x + \frac{b}{2a}\right)^2 = -\frac{(4ac - b^2)}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

⟸ The solutions of the equation, $ax^2 + bx + c = 0$



(Notice the "symmetry" to either side of $x = \frac{-b}{2a}$)