THE OTHER CULTURE: SCIENCE AND MATHEMATICS EDUCATION IN HONORS

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Chapter 9

ENGAGING THE HONORS STUDENT IN LOWER-DIVISION MATHEMATICS

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Mathematics in the Honors Curriculum

In 2010, the National Collegiate Honors Council published a set of twelve recommendations for fully developed honors colleges.[1] Those that refer to curriculum suggest that the program offer significant course opportunities across all four years of study, that the honors curriculum constitute at least twenty percent of a student's degree program, and that an honors thesis or project be required. Because mathematics is a core academic subject in most undergraduate degree programs, offering lower-division honors courses in mathematics is an appropriate means to increase the options students have for earning honors credits in courses that will be required for their degree program.

The lower-division honors courses in mathematics offered every year at the University of Texas at Arlington (UT Arlington) are an honors mathematics course for non-science majors and Honors Calculus I and II for science, technology, engineering, and mathematics majors. Common questions that arise about teaching honors mathematics courses include the following: What mathematics should be taught in a course for honors liberal arts majors, and how can it be taught to provide an honors experience? How do instructors make a calculus course honors? What should instructors expect from the students? After addressing characteristics and expectations of honors students, this article discusses ideas for creating meaningful honors experiences in lower-level honors courses in mathematics, both for non-science majors and for science majors. We consider first an honors course in mathematics for liberal arts majors and second, an honors course in calculus.

The Honors Student

In honors calculus the students are typically freshmen who are math or science majors. In contrast, students pursuing liberal arts majors may be taking their final mathematics course before graduation. In both cases, however, by designating themselves as honors students, they have invoked the following high expectations: Honors students should have the desire and motivation during their undergraduate years to educate themselves beyond the requirements of the degree they seek. They should want to understand what they study in great depth and within a broad context with a vision toward developing their career and becoming lifelong learners. Honors students should also be actively engaged in their learning and take ownership of their education; they should be in a class because they want to be there.

In new groups of honors students in lower-division mathematics courses, these qualities might not yet be well developed: the students might not be ready to actively learn on the first day of class. The qualities listed above must be taught and nurtured in the students as part of an honors education. Honors students, like other students, have heavy course loads and commitments outside of classes that may tempt them to not put enough time into homework, skip class occasionally, and not take the initiative on their own to excel to the best of their capabilities. Instructors of honors courses need to be aware of these pressures on the students and be armed with instructional strategies that will develop the qualities expected of them.

Expectations of an Honors Course in Mathematics

The following are goals for any honors course in mathematics. The institution offering these honors sections should carefully consider the appropriate maximum class size that will allow these goals to be accomplished effectively.

Ownership

Honors students should become the owners of the mathematics they study. They must take the initiative in deciding whether mathematical statements are true or false, whether a question is worthy of investigation or not, and how new mathematical concepts should be formally defined.

Communication

In an honors course, students should communicate and defend their arguments, both formally and informally and both orally and in writing, to the instructor and to their classmates.

Greater Maturity

An honors course should expand the students' view of what mathematics is and how to think about it so that, after completing the course,

students can look back and be amazed at how their mathematical maturity has developed.

Broader Context

An honors course in mathematics should give the students a perspective on how the subject has developed and how it is still evolving. This can include how other disciplines have influenced the development of mathematics or how mathematics has driven advances in other sciences or in the fields of mathematics.

Mathematics for the Honors Liberal Arts Student

Many students, even honors students, may enter mathematics courses with a fear of the subject. In UT Arlington's mathematics course designed for honors liberal arts students, entitled Honors Liberal Arts Mathematics, fear of mathematics is the invisible gorilla in the room at the beginning of the semester. The majors most represented in this class tend to be English and journalism, and, as a whole, the students do not project much confidence in their mathematical ability. The course is designed, however, to allow students to discover and explore topics in mathematics that may be completely foreign to them and may not even seem to be mathematics at first glance. Because of this structure, the students transcend many of their mathematical hang-ups and open their minds to the possibility of enjoying mathematics. The mathematical situations they study are often simple to state but incredibly rich in their depth. The students encounter and interact with mathematical areas that open problems that are understandable by novices. Faculty members expect these students to do mathematics that they initially believe to be far beyond their abilities, and it is amazing how they rise to the occasion.

Honors mathematics for liberal arts majors offers the opportunity to study all sorts of mathematics that are accessible to students at the college freshman level but have been omitted from the high school mathematics curriculum. Books used as sources for topics, discovery problems, and projects include *The Heart of Mathematics: An Invitation to Effective Thinking*, *To Infinity and Beyond*, and *Knots and Surfaces: A Guide to Discovering Mathematics*.[2] Readings are supplemented with articles from journals such as the *Mathematics Magazine*, the *American Mathematical Monthly*, and the *Notices of the AMS* and occasionally an interesting movie on mathematics. Some of the topics that can be included are graph theory, knot theory, the mathematics of voting, fair division, cryptography and coding theory, Fibonacci numbers, the Golden Rectangle, and notions of infinity.

The course also includes a few topics that the students have either studied or heard of, but it treats them in a new way. The following project is a good starting point to give the students a full flavor of the course.

Activity 1: The Pythagorean Theorem

After students are reminded of the famous Pythagorean Theorem, the first activity integrates two well-known geometric arrangements that provide visual ways of seeing why the Pythagorean Theorem is true. This geometric approach is taken in many textbooks, including *The Heart of Mathematics: An Invitation to Effective Thinking*[3] by Burger and Starbird (2010).

The Statement

This exercise begins with the class being asked to state the theorem and provide a few examples where the lengths of the sides are integers (Pythagorean triples). Students invariably recall studying the Pythagorean Theorem, and many of them are able to state it and use it correctly. The instructor can prompt the class to suggest finding such triples by listing the squares of the first fifteen or so positive integers and checking to see which two squares have a sum that is equal to another. The instructor then asks the class whether the theorem holds for right triangles where the lengths of one or more sides are not integers and, if so, to give some examples.

The Question

The challenge now comes when the class is asked: "How do you know that the Pythagorean Theorem is true? Is there a right triangle for which it does not work?" For many students, this moment may be the first time they have considered the question of *why* in mathematics. Now, not only are they confronting the question, but they must discover a solution themselves and defend their answers. The class as a whole should have a few minutes to think about this question. The purpose of this phase is for students to realize that throughout their study of mathematics, they have been using formulas without understanding why the formulas are true. They should now be curious to find an explanation for the Pythagorean Theorem.

Group Discovery

The students gather in groups of three to four around tables and work with cutouts to devise a geometric proof of the Pythagorean Theorem. The textbook by Burger and Starbird (2010) comes with a kit that contains cutouts of four identical right triangles and one square. These five shapes can be arranged in multiple ways. One possibility is as a large square whose edges are the hypotenuses of four right

triangles as in Figure 1. Another arrangement is as two concatenated squares as in Figure 2. The groups are asked to place their cut-outs, one set per group, on the table. For purposes of consistency in classroom conversation, the groups agree to denote the length of the long leg of each right triangle as **a**, the length of the short leg as **b**, and the length of the hypotenuse as **c.**

The assignment is to find two ways to arrange all five shapes to produce either one or two squares and to compute the area of each of these arrangements. After about 15 minutes, some groups will have found both configurations, perhaps aided by focused questions from the instructor. In calculating the areas, students will find that the area of the large single square, as in Figure 1, is c^2 and that the area of the concatenated squares, as in Figure 2, is $a^2 + b^2$.

Class Discussion

Finally, the instructor invites the students to explain how this exercise provides a justification of the Pythagorean Theorem. There are many alternative visual proofs of this theorem available in instructional materials. The benefits of this method are that the areas to be computed are for rectangular shapes and that algebraic manipulations are avoided, thus not frightening the students in the first activity. The two figures below are simple renditions of these geometric visualizations of the Pythagorean Theorem. Similar figures, in full color, may be found, for example, in Burger and Starbird (2010).

Figure 1: Geometric Visualization of the Pythagorean Theorem—1

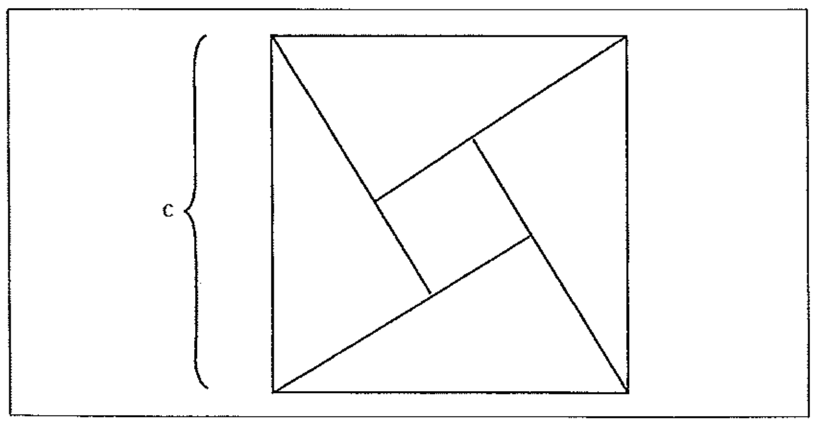
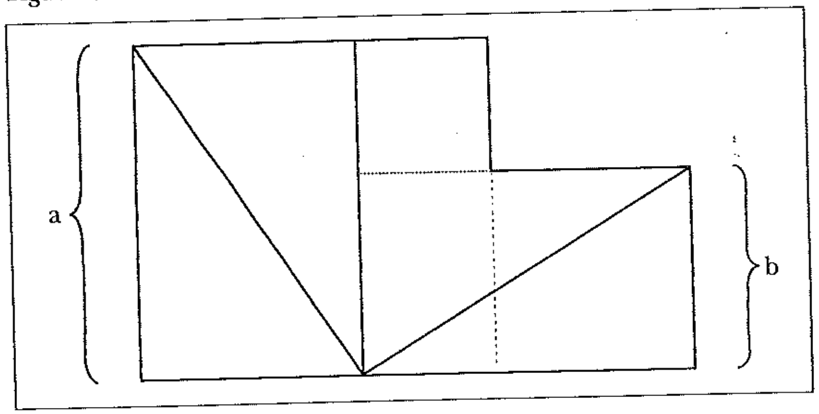


Figure 2: Geometric Visualization of the Pythagorean Theorem—2



During class discussion, the instructor should ask students to explain how they know that Figure 1 is actually a square. They should observe that the two angles at each corner are complementary, adding to 90º, and that the sides all have the same length. That Figure 2 is made up of two squares can be deduced from the first arrangement; the instructor should make sure that the class understands how to see this arrangement.

Writing Mathematics

To develop students' abilities to communicate mathematics accurately in writing, an assignment such as the following is a valuable exercise: Explain your proof of the Pythagorean Theorem, in writing, to a friend who may not clearly remember the theorem or be able to explain why it is true. (a) Give a clear statement of the Pythagorean Theorem and a few examples illustrating it. (b) Tell the reader that you are about to give a pictorial proof of why this theorem is true, and describe the set of cut-outs that you will use to do this. (c) Sketch the two arrangements of the cut-outs and explain how you can determine the area of each. (d) Explain how the Pythagorean Theorem emerges from this sketch.

Broader Context

After seeing that Pythagorean triples, such as (3, 4, 5) and (5, 12, 13), provide integer solutions of the equation $x^2 + y^2 = z^2$, the class will have an appreciation for the question of whether the corresponding equation, with the squares replaced by cubes, has integer solutions. The instructor can present the problem as a challenge exercise for homework: find a triple of integers (x, y, z) that satisfies $x^3 + y^3 = z^3$ and bring

it to the next class meeting. This can be assigned to one or two groups, and other groups can be asked to find a triple satisfying $x^n + y^n = z^n$ for other values of n>2.

Of course, the next class meeting may be disappointing if the number of solutions is small. This results will lead-in to a discussion of the history behind Fermat's Last Theorem and its eventual solution in 1992. A great way to conclude this study is by viewing *The Proof*, a 1997 NOVA production.[4] This documentary tells the story of Fermat's Last Theorem in a way that can inspire and excite liberal arts students about mathematics.

Other Variations

Students can prove the Pythagorean theorem by using other geometric configurations of squares and triangles. This assignment can serve as a bonus problem for interested students or as an end-of-class project.

For honors students, communication is rarely a problem, but communicating mathematics, especially for liberal arts majors, offers a new twist. One way in which students' mathematical confidence grows is in the realization that a mathematical argument does not need to consist of a two-column proof: it can be a convincingly rigorous prose argument. Honors students love to discuss ideas, and so they naturally build their understanding of the mathematics by verbalizing it. Hearing students, especially those who considered themselves math-phobic at the beginning of the semester, heatedly and reasonably arguing about mathematics is wonderful.

Activity 2 is an adaptation to a liberal arts honors setting of materials by Shipman that appear in Chapter 2 of *Active Learning Materials for Critical Thinking in a First Course in Real Analysis*[5] and in "Determining Definitions for Comparing Cardinalities,"[6] in which the author explains in more depth the mathematical ideas and teaching strategies that she used in creating and implementing these materials. Activity 2 illustrates how one can structure a discussion for honors liberal arts students about whether it is possible to take something away from a set and still have a set of the same cardinality. The discussions integrate questions and ideas from two activities in the references by Shipman cited above: *Relabeling doors: A dilemma in comparing quantities*, and *More circles or more squares?* Further activities on counting from these references that work nicely in an honors course for liberal arts majors are *Handing out cards* and *An orange tiger.*

Activity 2: The Counting Numbers and the Even Counting Numbers

The Context

Instructors can present this exercise before any discussion of finite or infinite cardinalities. Traditionally, before instructors introduce infinite cardinalities, they show students that when comparing two finite sets, they can determine that the sets have the same size by constructing a one-to-one pairing between them rather than counting each set and comparing the two numbers. Presenting the following exercise before any discussion of one-to-one correspondences gives honors students the opportunity to explore their own notions about counting and come up with their own arguments and ideas before being exposed to the ways that mathematicians have, after decades of work, agreed to understand counting.

The Question

Does the set of counting numbers {1, 2, 3, . . . }, which we denote by N, *contain a greater quantity of elements than the set of even counting numbers,* 2N = *{2, 4, 6, . . . }?* The students should offer their initial opinions so that they can discuss various points of view.

Initial Responses

The following are three common responses. (1) No, because both sets are infinite. (2) Yes, because N contains twice as many numbers as 2N. (3) It does not make sense to compare the sizes of infinite sets. The instructor will recognize a misconception about cardinality in each of these responses. The first correctly claims that N is not larger than 2N but incorrectly attributes this conclusion to the fact that both sets are infinite. The second response incorrectly assumes that a proper subset has a smaller cardinality than the original set. It also fails to recognize that the magnitudes of the numbers in a set has no influence on how many elements there are. The third response tries to avoid both dilemmas by claiming that comparing the sizes of two infinite sets is not reasonable. These three responses should be written on the board for the class to consider in the next step.

Group Discussion

The class may now discuss these options in groups of four. The assignment is for each person in the group to make a clear argument for or against each of the three responses and to present these arguments to the group. The students should base their arguments on clear mathematical reasons rather than on personal opinions or emotional inclinations. The goal is for each group to decide on one of the responses and, together, to formulate an argument to present to the

class. The group should also formulate a clear argument against each of the other two responses. If a group remains sharply divided, then the members should craft two dissenting opinions. All arguments, once agreed upon by the group, should be expressed clearly in writing. This part of the exercise may take up to twenty minutes.

Presentation of Arguments

Once all the groups have produced their written statements, each group should write the argument for its chosen position on the board under the statement it supports and give a brief explanation. Even if one group's explanation seems to mirror the argument that has already been recorded, the group should still write it down since slight changes in wording may have dramatic effects on meaning. After all the groups have recorded their statements, the counter-arguments can be presented and recorded, perhaps in a different color, under the supporting arguments for each statement.

Reflection

After hearing all arguments for and against each of the three initial responses, the students should quietly reflect for a few minutes on what they have heard and possibly modify their positions. Instructors will also need some time to reflect on what the class has said in order to make a logical transition to the next stage of the exercise. Usually the groups articulate one or two strongly stated arguments for and against each of the first two responses. Some key opinions that the students will have presented take into account *what* the numbers in each set are rather than focusing exclusively on *how many* there are. This observation is the motivation for the next sequence of questions.

Further Questions

Instructors can now direct the class as a whole to the following questions: Do the names of the people on a committee affect how many there are? Do the heights of the houses in a neighborhood affect how many there are? Do the sets {3, 6, 9, . . . , 300} and {5, 10, 15, . . . , 500} have the same size? How can you explain your answers? Students will generally agree about the answers to the first two questions : "No. What the items are does not affect how many there are." The students should then recognize without much help that constructing a one-to-one pairing between two sets in the third question is an intuitive and natural way to show that one set contains exactly as many elements as the other. Armed with this observation and a clear acceptance that what the members of a set are does not affect their quantity, the students are ready to accept as reasonable and correct the following definition.

Definition 1: Two nonempty collections have the same cardinality if there exists a one-to-one correspondence between them.

Instructors should ask the class to give examples of collections or sets that have the same cardinality according to this definition. Most students will see an obvious pairing between N and 2N (pairing a natural number k with $2k$) and conclude correctly that the sets have the same cardinality. It is instructive here to offer the counter-argument that 2N can be mapped into N by sending n to n. Does this contradict the definition? This discussion provides a good opportunity to emphasize the importance of carefully reading mathematical statements.

Checking Students' Confidence

The following true/false question tests students' acceptance of the perhaps unintuitive outcomes of Definition 1: *True or False? After all the work that mathematicians have put into comparing infinite cardinalities, it is still true that there is a greater quantity of counting numbers than even counting numbers.* Even after working through Activity 2 up to this point, some students may admit that they still secretly believe that there is a greater quantity of counting numbers than even counting numbers, despite openly acknowledging that the two sets have the same size by Definition 1. The activity *More Circles or More Squares?* in *Determining Definitions for Comparing Cardinalities* (Shipman, 2012) is designed to help students out of this uncomfortable position. The investigation that follows is a simplified version of this activity for liberal arts honors students. It will help them to dispel any lingering belief that the counting numbers should be more numerous than the even counting numbers.

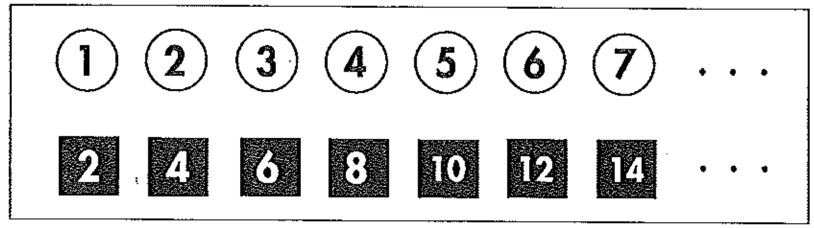
First, the class is divided into two groups. For the purpose of the exercise, Group 1 takes the position that the counting numbers are more numerous than the even counting numbers, and Group 2 takes the position that these two sets of numbers have the same size.

The students are asked to imagine that we have infinitely many solid circles and infinitely many solid squares (made of cardboard, for example). Each circle is white on one side and black on the other side. On the white side, the circles are numbered in black by the counting numbers. On the black side of each circle, the number $2k$ is written in white, where k is the number that appears on the opposite side. (For example, the circle with the number 5 written on its white side has the number 10 written on its black side.) The squares are colored and numbered in the same way.

Now imagine that the circles and squares are lined up on an infinitely long piece of glass. Figure 1 shows the view from the front of the

window, exhibiting the white sides of the circles and the black sides of the squares.

Figure 3: Front View (white sides of circles and black sides of squares)



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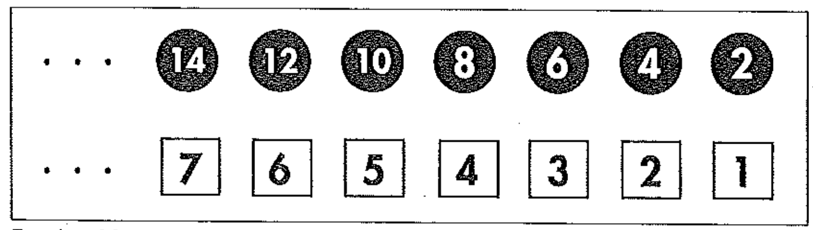
The students now think quietly about the following question before answering it:

Question: Judging by the position that your group has been asked to take, and looking at the numbers in figure, which are more numerous: the circles, the squares, or neither?

Group 1 will find the circles to be more numerous while Group 2 will conclude that the two sets have the same size.

The class then views the display from the back side and answers the question again:

Figure 4: Back View (black sides of circles and white sides of squares)



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From this perspective, Group 1 now finds the squares to be more numerous while Group 2 finds again that the two sets have the same size.

Conclusion

The class will see that the position of Group 1 (that the counting numbers are more numerous than the even counting numbers) gives contradictory answers, depending on which side of the window one is viewing, while the position of Group 2 (that the two sets have the same size) gives the same answer from both perspectives. This situation provides convincing grounds for liberal arts students to abandon the deceiving perception that the counting numbers are more numerous than even counting numbers and to accept Definition 1 as the "right" way to compare cardinalities.

A final phase of Activity 2 is to have the students decide whether comparing the sizes of infinite sets is an interesting question or whether it even makes sense to talk about it. Additional questions may be considered in a similar manner: Are some infinite sets larger than others? If more elements are added to an infinite set, must the set become larger? What do "larger" and "smaller" mean in terms of infinite sets?

Through discussions like these, instructors can lead students to formulate reasonable and self-consistent definitions of mathematical concepts. They will recognize that they have no alternative but to accept the often counterintuitive outcomes: there are exactly as many even integers as integers, and there are exactly as many numbers between 0 and 1 as there are on the whole infinite real line. Someone encountering such statements for the first time may think that mathematics is simply outlandish, but these honors students, from any major, can now offer solid explanations for why these outcomes, strange as they may seem, are indeed properties of infinite sets.

Two essays, "Cantor's New Look at the Infinite" and "To Infinity and Beyond" from the collection of essays in *To Infinity and Beyond,*[7] offer a historical perspective on the development of the mathematical concept of cardinality. In reading them, students are relieved to learn that the same questions that gave them headaches in class gave the most brilliant mathematicians at least as hard a time and caused at least as many arguments among them. Even Leopold Kronecker, Georg Cantor's mentor, refused to accept Cantor's rigorous and ingenuous formulation of cardinality. For more on the intriguing topic of infinity, Jorgensen and Shipman's *Limits of Infinite Processes for Liberal Arts Majors: Two Classic Examples*[8] offers engaging classroom activities that recast complex ideas on limits into settings that are tangible and visual, opening up new ways of mathematical thinking for the honors liberal arts student.

In Honors Liberal Arts Mathematics, students also learn to write mathematics with precision and clarity so that someone who does not already know the mathematics can read their work and understand the question, the method of solution, and the conclusion. Giving writing assignments to groups of two or three students allows them to check each other's writing for completeness and clarity. A writing assignment is usually based on a problem that has been discussed at length in class; thus the students have already explained their solutions verbally. The paper should include the following components.

Statement of Purpose

The introduction should state the purpose of the paper, which could be to present a mathematical problem, explain its solution, or verify that the solution works, and give examples to illustrate the results.

Statement of the Problem

This section should include, as appropriate, clearly labeled diagrams, definitions of concepts used in stating the problem, and a few illustrative examples.

Definitions

Before solving equations or analyzing a table or diagram, writers must tell their readers what the equations mean and what the entries and notations in the table or diagram are. This section should define all variables, with units if appropriate, and state the meaning of any equations, tables, or diagrams that will be considered.

Explanation of the Solution

The writers must take the readers carefully through each step of the solution because they must assume that the readers do not already know how to solve the problem.

Statement of the Result

After guiding readers through the solution and arriving at the final result, the writers should clearly state what has just been shown. This section will confirm that the arguments presented have indeed answered the problem posed at the beginning.

Verification of the Solution

Unless the problem is to prove a statement, checking the solution in some way is usually possible. This may be as simple as plugging in the answer to see that it works or it may involve verifying a strategy for winning a game by testing it on examples that cover a variety of possible cases.

In writing projects such as these, instructors have the opportunity to help the students, many of whom are English majors, to write precisely. What the students write should express what they mean, but often it does not. To help with this problem, one may assign shorter writing exercises in which students work in small groups to formulate in writing, without the help of the textbook or notes, a definition or mathematical result that has either been discussed previously in class or that they have seen in their previous mathematics education.

The task may be writing down what it means for a number to be prime or explaining clearly in writing how to put a fraction into lowest terms. Other challenges are to give a good definition of a function and to state in writing, without equations, what a circle is. Students can experiment with writing a clear algorithm to find the least common multiple or the greatest common divisor of two positive integers.

After each group has produced a written statement, the students should write them on the board. Each statement should be read carefully and taken at face value for exactly what it says, without interpreting it according to what the writer meant to say or according to what the readers believe that the writers meant. When a statement does not correctly express what is being defined, the instructor should present examples that satisfy what is written but do not correspond with what is intended. An example follows Statement A: To find the greatest common divisor of two positive integers, multiply the factors that go into both numbers. This claim means, then, that to find the greatest common divisor of 12 and 18, one must multiply 2, 3, and 6 to obtain 36. The students will readily agree that this statement is not correct but that this process is what the statement says to do. They might then modify this to Statement B: To find the greatest common divisor of two positive integers, multiply the prime factors that go into both integers. Now, to find the greatest common divisor of 24 and 36, Statement B instructs students to multiply 2 and 3 to obtain 6. The students will agree that this process is not correct either, and the discussion continues until the students write a correct statement.

After weeks of working together in this way, students will come to know each other well and will find peers with common interests and goals. The class often culminates in formal small group presentations on topics chosen by the groups. Some topics on which students have become the house experts include chaos, interconnections between mathematics and music, deciphering bar-codes, and the role of game theory in jury selection and medical decision-making.

Honors Calculus

The second major component of the lower-division honors experience in mathematics is Honors Calculus. A sequence of Honors Calculus courses is a wonderful opportunity to build a mathematical learning community among students. Many honors freshmen entering a college or university may not yet be sure what their career goals or major will be but will want to take enriched courses such as Honors Calculus. Indeed, this course may be for many honors students, of any potential major, a first experience in seeing, discovering, and understanding the richness of ideas that mathematics has to offer.

UT Arlington offers a year-long sequence, Calculus I and Calculus II, as honors courses. Roughly the same group of students takes both semesters of the course, so they receive an academic year of exposure to the same instructor and the same peers. Many of the students share common schedules in their other courses as well. Thus, the learning community fostered in their calculus course spills over and supports interactions in their other courses.

In a traditional calculus class, students spend much of the time understanding and practicing techniques for the computation of limits, derivatives, and integrals. As all mathematics instructors know, students can quite easily mistake the forest for the trees. In an honors course, instructors have the luxury of expecting and requiring the students to reflect upon and understand how the topics fit into their cumulative mathematical knowledge. All four primary expectations of an honors mathematics course can be developed in the context of an Honors Calculus sequence: ownership, communication, mathematical maturity, and broader context. While one instructor may choose a reform calculus text,[9] another may use a more standard text.[10] Shipman's "A Comparative Study of Definitions on Limit and Continuity"[11] is an additional resource well-suited for honors calculus; it takes a penetrating look at definitions on limit and continuity in a way that will prod students to think carefully about how every part of a definition is constructed.

The structure of the course rather than the textbook, however, provides the enrichment expected in an honors class. Each week of Honors Calculus consists of three hours of lecture and two hours of lab. Instead of conducting recitation in the traditional way by simply going over homework problems, instructors base each lab session on a worksheet of problems that they construct; these problems are much more challenging and far-reaching than those assigned from the textbook, and numerous resources for engaging projects, problems, and lab worksheets on topics in calculus are available.[12] In the labs, the students work in small, self-selected groups to find satisfactory solutions to the problems. The instructors serve as facilitators, posing questions that help the students gain a deep understanding of the problems but rarely answering questions. This approach serves many purposes. First, the students develop working relationships with the other students in class, relationships that will carry through to other courses that they will take together. Second, the students develop a sense of mathematical confidence. Because they become accustomed to instructors responding to all of their questions with more questions, they develop an ability to know when they are on the right track and begin to answer their own leading questions. This lab structure is modeled in part after the format utilized by the Emerging Scholars Program at the University of Texas at Austin and the MathExcel Program at the University of Nebraska-Lincoln. Epperson's seminal and widely disseminated set of worksheets[13] for these programs served as a catalyst for worksheets developed for this setting. To emphasize accountability, all students must submit their solutions to the lab worksheets the following week.

In one type of problem considered in the lab sessions, the students play the role of the teacher in that they need to create problems or find examples fitting specifications to illustrate key ideas. By building their own examples rather than simply applying theorems or results derived in class, the honors students develop a more complete understanding of the concepts and an ownership of those ideas. The following example is typical of the problems found in the reform calculus text or the worksheets referred to above.

Example 1: A Sum Rule for Non-Existent Limits?

The Problem

Do there exist two functions $f(x)$ and $g(x)$ and a constant c such that $\lim_{x\to c} f(x)$ *does not exist,* $\lim_{x\to c} g(x)$ *does not exist, but* $\lim_{x\to c} (f(x) + g(x))$ *does exist? Either find an example, or prove that no such example exists.*

To solve this problem, students must know more than how to compute a limit: the students must put their knowledge of functions into the context of limits to try to create a counterexample to a common mistaken theorem that calculus students often try to apply. First, they must think about how to build a function for which the limit does not exist at some value $x = c$. Then they need to consider how they can fix the bad behavior by adding another function. This problem can be adapted or expanded by considering a different limit rule, e.g., the limit of the product of functions or the limit of the composition of functions, and also by changing the limit behavior of the constituent functions. There are infinitely many solutions to this problem, but one simple solution is to let $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$ and consider their behavior at $x = 0$.

A second type of problem emphasized on the lab worksheets asks students to explain concepts in their own words: *Give an explanation of the Mean Value Theorem that a pre-calculus student could understand. Your explanation should be both verbal and pictorial.* These may also be interpretation problems explaining what a derivative means. The reform calculus textbook *Calculus: Single Variable* by Hughes-Hallett et al. offers many problems of this kind. One such example taken from the Hughes-Hallett text is the following:

Example 2: A Derivative in Practical Terms

The Problem

A company's revenue from computer sales, R, measured in thousands of dollars, is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $R = f(a)$. *Explain what the statement* $f'(301) = 2$ *means in practical terms.*

It is important, especially on these interpretation problems, to require the students to really answer the question. Students will often solve the previous problem by saying that $f'(301) = 2$ means the derivative of f at 301 is 2. The instructor's job is to draw out the answer by asking a sequence of scaffolding questions that guide the groups to fully consider the problem. For example, what are the units of the number 301? What are the units of the number 2? What does f measure? Do you have any information about the value of $f(301)$? Do you need it? How would the computer company use information about f'? If the company is already spending $301,000 on advertising, would it be wise for the company to increase its advertising expenditures? If, $f'(301) = 0.3$, would your answer to the previous question change?

A complete answer to this problem should include the following information: $f'(301) = 2$ means that if the company is already spending $301,000 on advertising and it spends a little bit more on advertising, it would expect its revenue to increase by approximately twice the amount of increase in advertising expenditure. For instance, if it spends $301,100 on advertising, it would expect its revenue to go up by about $200, so it would make back the extra $100 it spent on advertising, plus $100 more.

These problems seek to make connections between the computational and conceptual ideas of calculus and hone the honors students' abilities to communicate and to justify their perceptions. Instructors should expect students to periodically present their ideas and solutions to problems to the rest of the class, further reinforcing the need to communicate mathematics. To encourage the students to keep up with the regular homework assignments, one lab each week starts with ten to twenty minutes of presentations of homework problems by students. Instructors can select three homework problems assigned the previous week. The students do not know ahead of time which problems will be presented. Three randomly selected students present their homework problems. During the semester, each student will have the opportunity to present.

The following grading scheme has worked well. If students are not at lab that day, they earn zero points for the presentation. If students are at lab but do not feel ready or willing to present the requested problem, they earn two points for attendance. Five points are awarded if a student presents a solution. The atmosphere during the presentations should not be stressful or pressured. They are opportunities for the students to practice communicating mathematics and refining their solutions. Again, emphasizing that mathematics needs to be communicated and that students who present their work to others often find mistakes is important. This practice mirrors the way professional mathematicians work and make discoveries.

A major component of the Honors Calculus I and Calculus II courses is an extensive collaborative project culminating in a professionally written report. The students work in small groups of four, and most of their work on this project is done outside of class. The groups are chosen by the instructor to ensure that each group contains students with different abilities and that the students' schedules align for at least two available working hours each week. All the groups work on the same project. The students have a month and a half to complete the project, and the final product must demonstrate high mathematical quality and be well written.

Crucial to the success of these group projects is constructing a timeline for completion. A minimal timeline should include a date for an initial meeting with the instructor to discuss the group's preliminary ideas, the date by which the first draft is due to be submitted and reviewed, and the final submission date. Each group submits a single final paper, which should be mathematically typeset and include appropriate diagrams. Students should be encouraged to address their paper to a reader who is superior to them in position, a supervisor for instance, but equal to them in knowledge of calculus. In addition, their paper should be rich in context, explanation, and prose. Often students expect that a mathematical paper should look like the solution to a homework problem with nothing but numbers and symbols. Presenting the students with a model project write-up from a previous semester or a sample of exemplary mathematical writing at the college level is helpful.

Many sources offer rich calculus project problems.[14] A recent Calculus I group project, for example, was based on designing a suspension bridge to satisfy prescribed dimensions. The students needed to determine the length of a catenary supporting the bridge, using only their knowledge of Calculus I. They had not yet encountered the arc length formula, and through the project, the students developed the formula. Another Calculus I project involved measuring the volume of wine in a barrel with a bung rod after finding the optimal barrel dimensions.[15] A recent Calculus II project had the groups finding the generating function of the Fibonacci sequence via Taylor series, and another revolved around employing power series and clever use of trigonometric identities to find more efficient ways to compute many digits of π. Such group projects reinforce the cooperative atmosphere of the classroom, requiring the students to work together extensively on their own time. The final papers the groups produce are often creative and entertaining to read, which is always a delight for the instructors.

Anecdotal evidence of the successful creation of an atmosphere of enjoyable community learning occurred during the final exam period for the Honors Calculus I class in Fall 2006. Lying in wait to celebrate their final exam by blowing bubbles, the students surprised their professor by decorating a room with streamers and confetti and providing a buffet of home-baked cupcakes. The students truly feel that the class is their own and that their peers are their collaborators, defining a true honors experience in mathematics that will likely remain a highlight for the students for the rest of their undergraduate careers.

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Notes

[1]<http://www.nchchonors.org/faculty-directors/basic-characteristics-of-a-fully-developed-honors-college/> Accessed 10 April 2012.

[2]Edward B. Burger and Michael P. Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking,* 3rd ed. (Everyville, CA: Key College Publishing, 2005); Eli Maor, *To Infinity and Beyond* (Princeton University Press, 1987); David W. Farmer and Theodore B. Stanford, *Knots and Surfaces: A Guide to Discovering Mathematic* (Providence: American Mathematical Society, 1996).

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[12]For a repository of projects and ideas, see Caren L. Diefenderfer and Roger B. Nelsen, eds. *The Calculus Collection: A Resource for AP and Beyond* (The Mathematical Association of America, 2010). For a seminal

set of worksheets, see James Epperson, "Calculus I Worksheets" and "Calculus II Worksheets," from Treisman Workshop Resources Worksheet Archive. Available online at <http://math.sfsu.edu/hsu/workshops/resources.html>.

[13]James Epperson, "Calculus I Worksheets" and "Calculus II Worksheets" from Treisman Workshop Resources Worksheet Archive. Available online at <http://math.sfsu.edu/hsu/workshops/resources.html>.

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Chapter 10

STATISTICS IN HONORS: TEACHING STUDENTS TO SEPARATE TRUTH FROM "DAMNED LIES"

LISA W. KAY

Introduction

H. G. Wells reportedly once said, "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write."[1] People need to have a basic understanding of fundamental statistical concepts because they are constantly bombarded with quantitative information in the media. In many cases everyday activities like reading the newspaper require statistical thinking if people are to process the given information intelligently. Understanding the risks associated with taking a new medication, for example, requires a rudimentary knowledge of probability.

A few years ago, the author proposed a statistics course as a junior-level elective for the Eastern Kentucky University (EKU) Honors Program because statistical thinking is particularly crucial for honors students. They need to be able to use quantitative information to support or refute arguments, and honors students often need statistics in their theses. At times senior thesis presentations contain statistical atrocities: the use of voluntary response samples, which produce biased results; the inclusion of flawed graphical depictions of data; and the misuse of statistical terminology. A statistics course in honors offered at the junior level should reduce the number of such errors in senior work.

Design of the Course

As an alumna of the program, the author knew that she wanted to propose a course that features some of the program's hallmarks: an interdisciplinary theme, team teaching, and panel presentations. In addition, the course should include the recommended characteristics found in the Guidelines for Assessment and Instruction in Statistics Education (GAISE) <http://www.amstat.org/education/gaise/>: